13.2 Calculus on 3D Curves

Example: Consider

$$x = t, y = 2 - t^2$$

which can also be written as

$$\boldsymbol{r}(t) = \langle t, 2 - t^2 \rangle$$

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
   What do these represent?

2. When $t = 1$, what is the location and what is the slope of the tangent line?

3. Find a vector in the direction of the tangent line at $t = 1$.

$\boxed{1}$ $\frac{dx}{dt} = 1$ = horizontal velocity

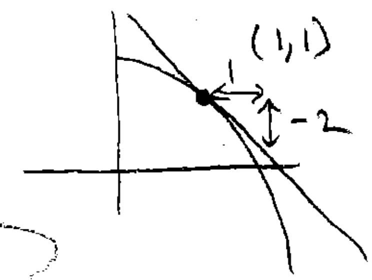
$\frac{dy}{dt} = -2t$ = vertical velocity

$\boxed{2}$ $t = 1 \Rightarrow x = 1, \; y = 1$

$x'(1) = 1, \quad y'(1) = -2$

RECALL

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2}{1}$$



$y = -2(x - 1) + 1$

MATH 124

$\boxed{3}$ MATH 126

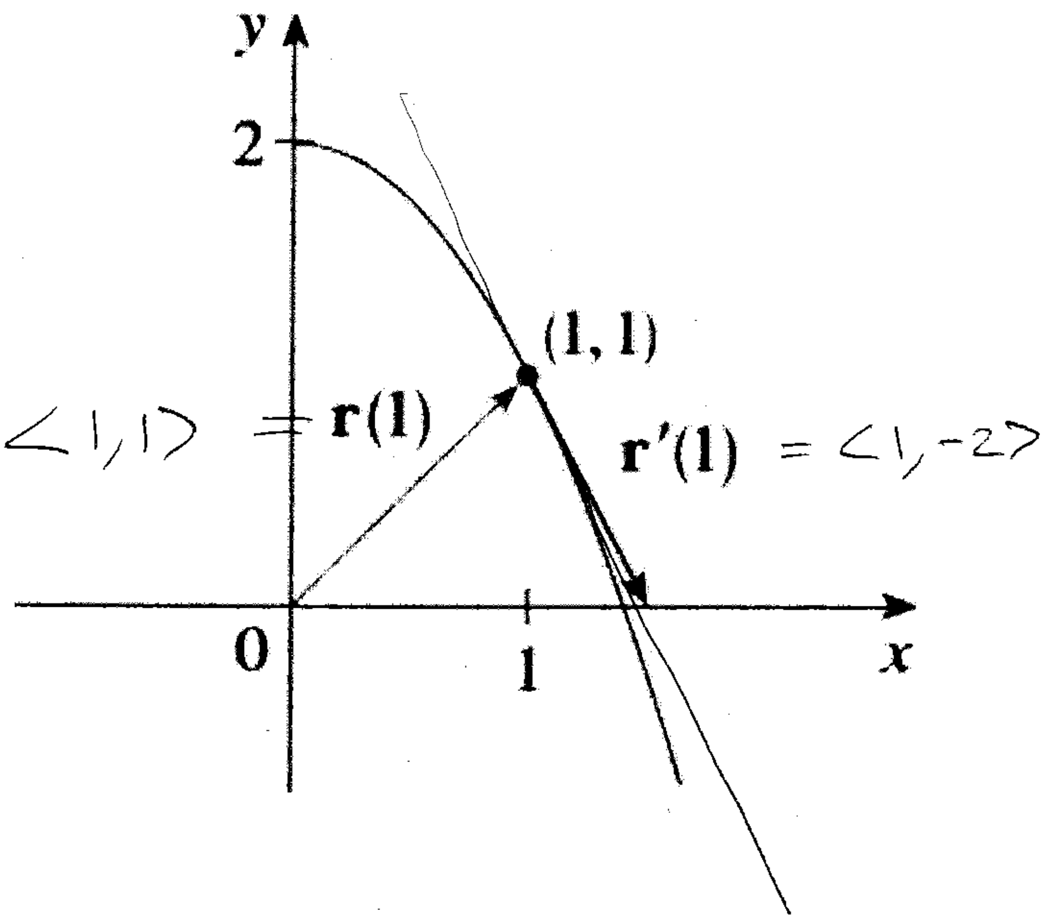
$\vec{r}_0 = \langle 1, 1 \rangle$

$\vec{v} = \langle 1, -2 \rangle$ = TANGENT VECTOR

$$\boxed{\begin{aligned} x &= 1 + 1t \\ y &= 1 + -2t \end{aligned}}$$

Visual of last example:

$$\boldsymbol{r}(t) = \langle t, 2 - t^2 \rangle$$



$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 1 - 2t \end{aligned}}$$

TANGENT LINE

In general: Vector Calculus

For $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we define

$\vec{r}(t+h) - \vec{r}(t)$

$$\vec{r}\,'(t) = \lim_{h\to 0}\left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \right\rangle$$

which is the same as

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$$

And

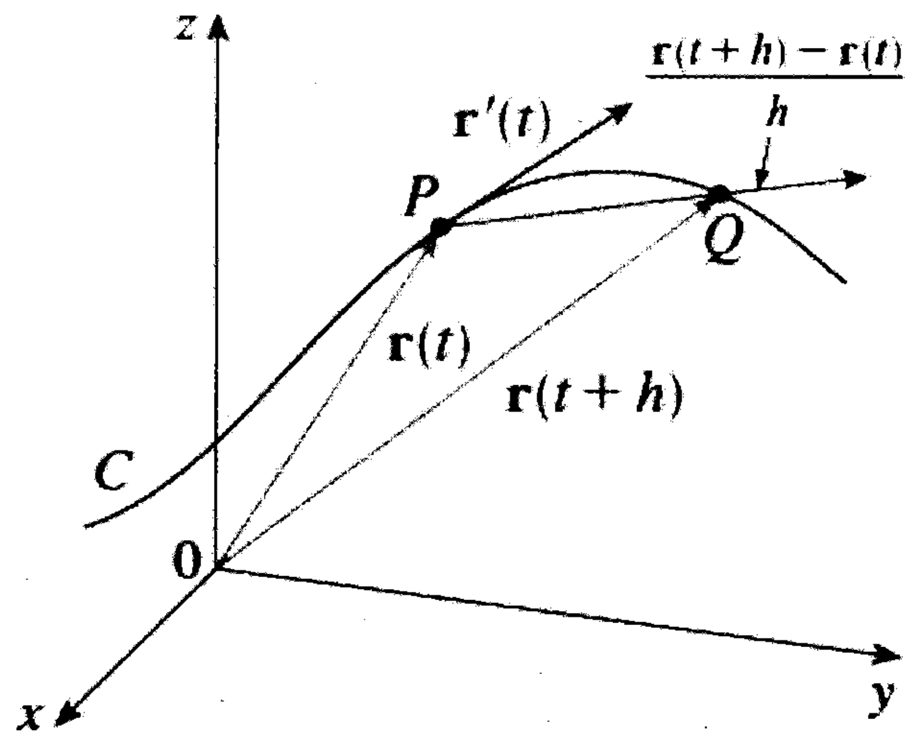
$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$$

is a tangent vector to the curve.

Do calculus **component-wise**!

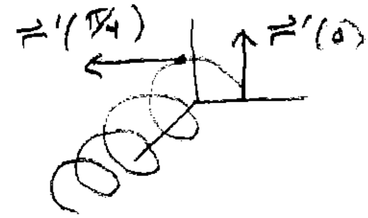
NOTE: ① $\overrightarrow{PQ} = \vec{r}(t+h) - \vec{r}(t)$

② DIVIDING BY h, RESCALES BUT KEEPS DIRECTION THE SAME.



Example

$\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle.$



1. Find $\vec{r}\,'(t)$
2. Find $\vec{r}\,'(0)$ and $\vec{r}\,'(\pi/4)$
3. Find equations for the tangent line at $t = 0$.
4. Find equation for the tangent line at $t = \pi/4$

$\boxed{1}$ $\vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$

$\boxed{2}$ $\vec{r}\,'(0) = \langle 1, 0, 2 \rangle$ $\Leftarrow$ PARALLEL TO XZ-PLANE

$\vec{r}\,'(\pi/4) = \langle 1, -2, 0 \rangle$ $\Leftarrow$ PARALLEL TO xy-PLANE (HIGH POINT)

$\boxed{3}$

$\vec{r}(0) = \langle 0, 1, 0 \rangle$

$\vec{r}\,'(0) = \langle 1, 0, 2 \rangle$

$$\boxed{\begin{aligned} x &= 0 + 1\,t \\ y &= 1 + 0\,t \\ z &= 0 + 2\,t \end{aligned}}$$

$\boxed{4}$

$\vec{r}(\pi/4) = \langle \pi/4, 0, 1 \rangle$

$\vec{r}\,'(\pi/4) = \langle 1, -2, 0 \rangle$

$$\boxed{\begin{aligned} x &= \pi/4 + 1\,t \\ y &= 0 + -2\,t \\ z &= 1 + 0\,t \end{aligned}}$$

Summary of 3D calculus

$\vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ *tangent vector* (13.2)

$\int \vec{\boldsymbol{r}}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$ antiderivative vector (13.2/4)

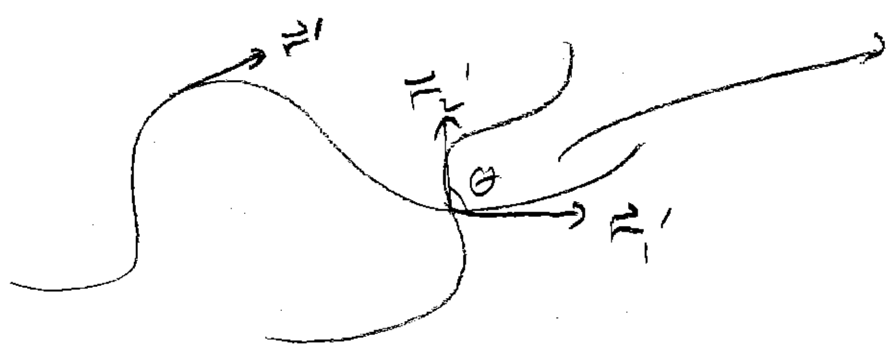
13.3 Curvature, Arc Length, Normal Vector

13.4 Velocity, Speed, Acceleration (components of acceleration)

$\vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ *velocity vector* (13.4)

$|\vec{\boldsymbol{r}}'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}$ *speed (13.4)*

$\vec{\boldsymbol{r}}''(t) = \langle x''(t), y''(t), z''(t)\rangle$ *acceleration vector* (13.4)



ANGLE OF INTERSECTION

$\Leftrightarrow$ ANGLE BETWEEN TANGENT VECTORS